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Cubic Interpolated Pseudo-particle Method (CIP) for Solving Hyperbolic-Type Equations

H. TAKEWAKI, A. NISHIGUCHI, AND T. YABE

Institute of Laser Engineering, Osaka University, Suita, Osaka 565, Japan

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A new cubic-polynomial interpolation method, where the gradient of the quantity is a free parameter, is proposed for solving hyperbolic-type equations. Various choices of the gradient are investigated, and a stable and less diffusive scheme is made possible without the clipping or the flux-correction procedure. © 1985 Academic Press, Inc.

I. INTRODUCTION

At present, there are many techniques for solving the hyperbolic-type equations. To avoid numerical instability and diffusion, various techniques are employed: two of these are the numerical viscosity [1] and the FCT algorithm [2].

In treating the advection terms, a profile within a mesh must be correctly described, otherwise a large number of meshes must be used. There are mainly two ways to reduce the number of meshes. One is to use a rezoning of mesh to follow the variation of quantities. The other is to approximately describe a profile within a mesh. The former has been extensively studied by many authors [3] and proves to be satisfactory in a one-dimensional problem, but its ability deteriorates in a multi-dimensional situation. The latter way includes a variety of methods: the finite element method (FEM), the boundary element method (BEM), the particle scheme, and so on. The FEM and BEM have not always worked successfully for hyperbolic-type equations. An ordinary particle scheme, such as PIC [4], employs a number of particles to describe the profile within a mesh. In order to reduce the number of particles and attain a higher-order accuracy in the advection, we proposed the second-order accurate fluid particle scheme (SOAP) in our previous paper [5, 6]. The scheme assumes a distribution of physical quantities within a finite-sized particle so that physical quantities are exactly conserved.

In this paper, we propose a new method (CIP) to solve the hyperbolic-type equations. The method takes a similar approach to the FEM, the BEM, and the particle scheme, and is interpreted to be an extension of a one-particle version of our SOAP scheme. Accordingly the scheme may be a bridge between the FEM and the particle schemes.

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In Section II, the basic algorithm is introduced. Section III provides the results of test runs with some modifications and discussions. From these results, it is proven that a stable and less diffusive scheme is possible by the combination of two different schemes. The CIP scheme gives better results than the shape-preserving spline [7] and Knorr and Mond's results [8] for square-wave and sine-wave propagations.

II. Basic Algorithm

At first, let us consider a simple model equation such as

$$\frac{\partial f}{\partial t}+c\frac{\partial f}{\partial x}=0, \tag{1}$$

where c is a constant value. Equation (1) is integrated over an interval $(x_{i-1/2}, x_{i+1/2})$, (t_n, t_{n+1}) to be

$$\int_{x_{i-1/2}}^{x_{i+1/2}} (f^{n+1}(x)-f^n(x))\,dx = -c\int_{t_n}^{t_{n+1}} (f_{i+1/2}(t)-f_{i-1/2}(t))\,dt, \tag{2}$$

where the superscript n and the subscript i on f mean the value of f at $t=t_n$ and $x=x_i$, respectively. Since the profile propagates at a speed c, the right-hand side in Eq. (2) can be written as

$$\begin{aligned}\text{r.h.s.} &= -\int_{x_{i+1/2}-c\Delta t}^{x_{i+1/2}} f^n(x)\,dx+\int_{x_{i-1/2}-c\Delta t}^{x_{i-1/2}} f^n(x)\,dx \\ &\equiv -\Delta F_i+\Delta F_{i-1}\end{aligned} \tag{2'}$$

as seen in Fig. 1. Here $\Delta t=t_{n+1}-t_n$. If the spatial profile of f^n is known, all terms except the term of f^{n+1} in Eq. (2) can be obtained from the integration of f^n.

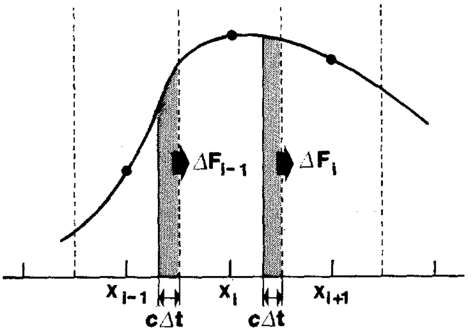

FIG. 1. The schematics of the flux calculation.

There are many choices for an approximate functional form of f. In this paper, we choose the cubic-polynomial interpolation within an interval (x_{i-1}, x_i) such as

$$f(x) = a_i(x - x_{i-1})^3 + b_i(x - x_{i-1})^2 + f'_{i-1}(x - x_{i-1}) + f_{i-1}. \tag{3}$$

Here f_{i-1} and f'_{i-1} are the value and the spatial derivative of f at $x = x_{i-1}$, respectively. If we require the continuities of f and f' at all boundaries $x_1, ..., x_{i-1}, x_i, x_{i+1}, ...$, then we obtain

$$a_i \Delta x^3 + b_i \Delta x^2 + f'_{i-1} \Delta x + f_{i-1} = f_i \tag{4a}$$

$$3a_i \Delta x^2 + 2b_i \Delta x + f'_{i-1} = f'_i, \tag{4b}$$

where $\Delta x = x_i - x_{i-1} = \text{const}$. Consequently, the coefficients a_i and b_i can be described in terms of f and f' as

$$a_i = \frac{(f'_i + f'_{i-1})\, \Delta x - 2(f_i - f_{i-1})}{\Delta x^3} \tag{5a}$$

$$b_i = \frac{3(f_i - f_{i-1}) - (f'_i + 2f'_{i-1})\, \Delta x}{\Delta x^2}. \tag{5b}$$

Once the functional form of f is given, Eqs. (2) and (2′) lead to

$$\begin{aligned} &1/192(18f^{n+1}_{i+1} + 156f^{n+1}_i + 18f^{n+1}_{i-1} - 5f'^{n+1}_{i+1}\, \Delta x + 5f'^{n+1}_{i-1}\, \Delta x) \\ &\quad = 1/192(18f^n_{i+1} + 156f^n_i + 18f^n_{i-1} - 5f'^n_{i+1}\, \Delta x + 5f'^n_{i-1}\, \Delta x) \\ &\quad\quad -(\Delta F_i - \Delta F_{i-1})/\Delta x \end{aligned} \tag{6a}$$

and

$$\begin{aligned} \Delta F_i = {} & (-\kappa/8 + \kappa^2/8 + \kappa^3/6 - \kappa^4/4) f'^n_{i+1}\, \Delta x^2 \\ & + (\kappa/8 + \kappa^2/8 - \kappa^3/6 - \kappa^4/4) f'^n_i\, \Delta x^2 \\ & + (\kappa/2 - 3\kappa^2/4 + \kappa^4/2) f^n_{i+1}\, \Delta x \\ & + (\kappa/2 + 3\kappa^2/4 - \kappa^4/2) f^n_i\, \Delta x, \end{aligned} \tag{6b}$$

where $\kappa = c\Delta t/\Delta x$.

When all f^n_i and f'^n_i are given, the value of the right-hand side of Eq. (6a) is calculated. But both f^{n+1} and f'^{n+1} cannot be calculated at once from this equation alone. Then f'^{n+1} must be determined by the other method. For this purpose, the spatial derivative of Eq. (1),

$$\frac{\partial f'}{\partial t} = -c\frac{\partial f'}{\partial x}, \tag{1'}$$

is used.

In the linear case given in Eq. (1), Eq. (1′) refers to the propagation of the gradient and this solution can be used for estimating the spatial derivative $f_i'^{n+1}$ as

$$f_i'^{n+1} \equiv f'(x_i, t_{n+1}) = f'(x_i - c\Delta t, t_n). \tag{7}$$

For more general equations, the method can be easily extended. For example, if Eq. (1) has an external force term such as

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}, \tag{1a}$$

the gradient of f is advanced at first by the equation

$$\frac{\partial f'}{\partial t} = \frac{\partial^2 g}{\partial x^2} \tag{1a′}$$

and then it is shifted according to Eq. (7). Here, Eq. (1a′) is solved by the finite difference approximation. Once f'^{n+1} is given by the above method, f^{n+1} can be obtained by solving Eq. (6a).

III. Linear Wave Propagation

The reason why the cubic-polynomial interpolation is employed in our scheme is clarified in view of the basic algorithm mentioned in the previous section. The continuities of the value and the first spatial derivative are fulfilled with one free parameter retained. The free parameter, which is f'^{n+1} as in Eq. (7), gives us a variety of choices for the scheme and allows us to find a stabler and less diffusive scheme.

It is not necessary to adopt the above-mentioned method for the estimation of f'^{n+1}. Some of the possibilities for f'^{n+1} are described and tested in this section. For this purpose, we use the problem of square-wave propagation described by Eq. (1) with $\kappa = c\Delta t/\Delta x = 0.2$, where Δt is the time step and Δx is the uniform mesh size. It should be noted that our scheme can be easily modified for a nonuniform mesh size such as SOAP and FEM. In the following, various choices for f'^{n+1} are described and the results are shown in Fig. 2 at 1000 time steps after.

(a) *Shifted Gradients.* The shifted gradient given by Eq. (7) is used.

(b) *Minor Gradients.* The gradient of f is estimated by the finite difference of the values f at mesh points. If the one-sided finite difference is used, there exist two choices—upstream and downstream. Here, we adopt the one which has a smaller absolute value:

$$f_i'^{n+1} = \begin{cases} (f_{i+1}^n - f_i^n)/\Delta x & \text{if } |f_{i+1}^n - f_i^n| < |f_i^n - f_{i-1}^n| \\ (f_i^n - f_{i-1}^n)/\Delta x & \text{otherwise.} \end{cases} \tag{8a}$$

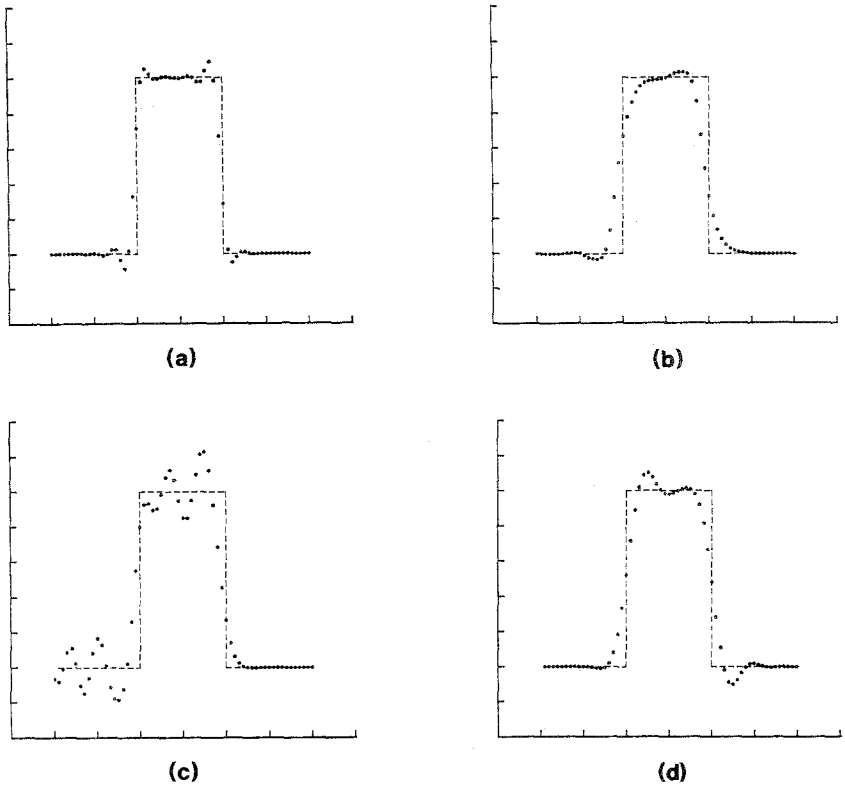

FIG. 2. Propagation of a square wave for $\kappa = c\Delta t/\Delta x = 0.2$ after 1000 time steps in the case of (a) shifted-gradient, (b) minor-gradient, (c) average gradient, and (d) upstream gradient. The dashed line indicates the analytic solution.

(c) *Average Gradients.* As in case (b), the finite difference is used. But the centered difference is used here:

$$f_i'^{n+1} = (f_{i+1}^n - f_{i-1}^n)/2\Delta x. \tag{8b}$$

(d) *Upstream Gradients.* Here the upstream difference is used.

$$f_i'^{n+1} = \begin{cases} (f_i^n - f_{i-1}^n)/\Delta x & \text{if} \quad c > 0 \\ (f_{i+1}^n - f_i^n)/\Delta x & \text{otherwise.} \end{cases} \tag{8c}$$

The results of all cases are shown in Figs. 2(a)–(d) corresponding to cases (a)–(d), respectively. In view of these results, the acceptable schemes are cases (a) and (b). If we use the clipping as used in the FCT algorithm or other methods to eliminate the humps, case (a) is the most hopeful scheme because of its small numerical diffusion. But it is shown in the following explanation that this overshooting will disappear by a simple method. The answer to this problem is given by considering the origin of the overshooting.

The overshooting occurs in case (a) because the initial discontinuities at the square wave's edges are so large that the cubic polynomial for interpolation cannot trace the wave. Subsequently, the overshooting occurs only after one time step in order to adjust itself. Once the profile is adjusted to the polynomials, the humps at the edges scarcely grow. If the initial profile is taken to be as smooth as possible, then the problem is solved. In general, such an initial setting is hardly acceptable. Accordingly, we take a simpler way. We initially broaden the discontinuities of the

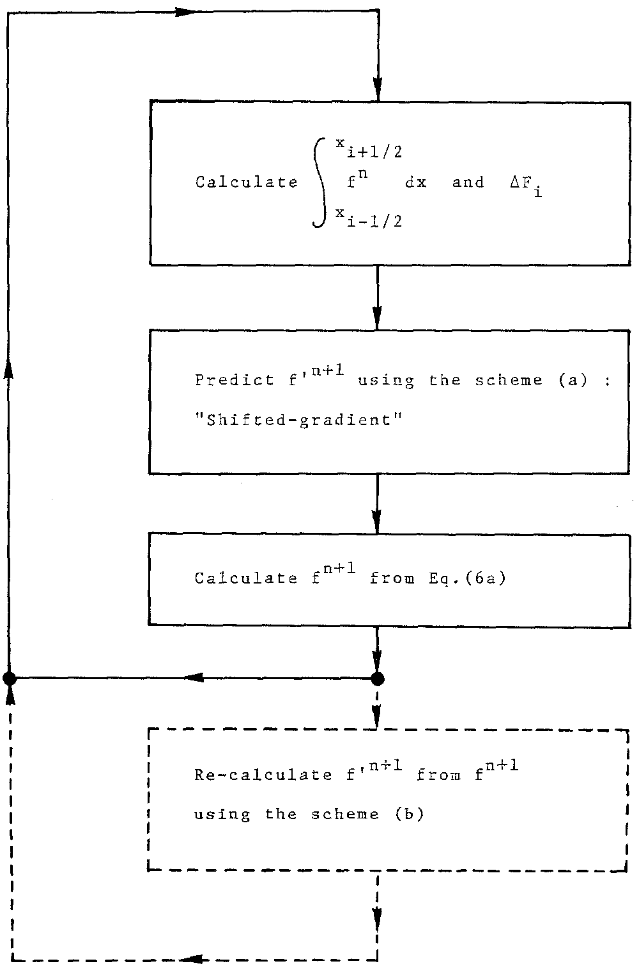

FIG. 3. The flow chart of the CIP scheme. The part shown by the solid line is calculated at all time steps and the part shown by the dashed line is inserted during the initial 10 steps and thereafter every 50 steps.

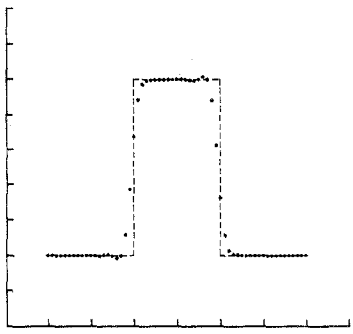

FIG. 4. Same run as in Fig. 2, but the algorithm (CIP) given in Fig. 3 is used.

square wave, coupling scheme (a) with scheme (b) as shown in Fig. 3 and when the profile becomes smooth enough for the cubic polynomial to trace it naturally, we switch the algorithm to (a). In some cases, this procedure alone is not enough for the elimination of the overshooting which appears at a later time. Overshooting of this kind is due to the accumulation of errors. Because the gradient of the profile is shifted by an approximate procedure [Eq. (7)], the accumulated errors cause the mismatch between the profile and the predicted gradient. In order to avoid the growth of overshooting, operation (b) is sometimes inserted as shown in Fig. 3. The example given in Fig. 4 is obtained by the following procedure: Scheme (b) is inserted to recalculate the gradient f'^{n+1} from f^{n+1} obtained by scheme (a) at the initial 10 steps. After this only scheme (a) is used. After every 50 time steps, scheme (b) is inserted. Combination of the two schemes gives us a new scheme which has less diffusion and less overshooting. We call the scheme CIP (cubic-interpolated pseudo-particle), because the procedure is basically similar to the particle scheme SOAP as will be shown in a forthcoming paper.

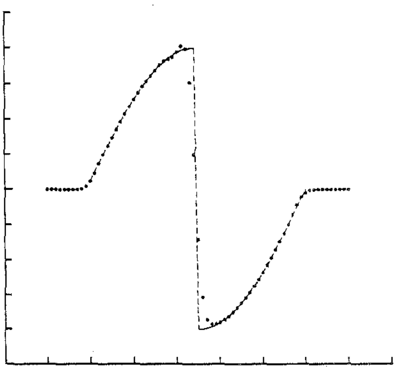

FIG. 5. Propagation of a sine wave with discontinuity for $\kappa = c\Delta t/\Delta x = 0.2$ after 1000 time steps. The algorithm (CIP) given in Fig. 3 is used.

Figure 5 shows the result by the CIP in which a sine wave with discontinuity is being propagated. The discontinuity is represented by four points and the amplitude of the positive peak increases by about 1.11 % and that of the negative peak decreases by about 3.33 % after 1000 time steps.

The results given in Figs. 4 and 5 are less diffusive than the shape-preserving spline [7] and Knorr and Mond's results [8].

IV. Conclusion

In this paper, we proposed a new cubic-polynomial interpolation scheme, where the gradient of the quantity is a free parameter. The choice of the gradient has proved to be important. The gradient is calculated by the spatial derivative of the model equation and is sometimes corrected by using the value of quantity in order to eliminate the mismatch between the value and the predicted gradient. This scheme can be extended to nonlinear equations. The result will be given in a forthcoming paper.

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References

1. R. Landshoff, "A Numerical Method for Treating Fluid Flow in the Presence of Shocks," Los Alamos Scientific Laboratory Report LA–1930, Los Alamos, N. M., 1955; M. L. Wilkins, *J. Comput. Phys.* **36** (1980), 281; J. Von Neumann and R. D. Rihtmyer, *J. Appl. Phys.* **21** (1950), 232.
2. J. P. Boris and D. L. Book, *J. Comput. Phys.* **11** (1973), 38.
3. J. F. Thompson and Z. U. A. Warsi, *J. Comput. Phys.* **47** (1982), 1.
4. M. W. Evans and F. H. Harlow, "The Particle-in-Cell Method for Hydrodynamics Calculations," Los Alamos Scientific Laboratory Report LA–2139, Los Alamos, N.M., 1957; F. H. Harlow, *in* "Methods in Computational Physics," Vol. 3, Academic Press, New York, 1964; A. A. Amsden, "The Particle-in-Cell Method for the Calculation of the Dynamic of Compressive Fluids," Los Alamos Scientific Laboratory Report LA–3466, Los Alamos, N.M., 1966.
5. A. Nishiguchi and T. Yabe, *J. Comput. Phys.* **47** (1982), 297.
6. A. Nishiguchi and T. Yabe, *J. Comput Phys.* **52** (1983), 390.
7. M. M. Shoucri, *J. Comput. Phys.* **49** (1983), 334.
8. G. Knorr and M. Mond, *J. Comput. Phys.* **38** (1980), 212.